feature

Say Crease!

Folding Paper in Half

Miles Please

When I was a small child I read the following claim about paper folding: "It is impossible to fold any piece of paper in half more than eight times no matter how big, small, thin or thick the paper is" ([1], pp. 32-43). At that time I accepted the fact after some experimentation. But last month, to my amazement, I discovered that a grade 11 student in California, Britney C. Gallivan, had mathematically disproved the above statement in 2002 [1]. This article discusses the proof given by Gallivan regarding "folding paper in half, n times".

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Introducing the Problem: Gallivan's Rules

We first state the rules to be followed while folding a sheet of paper in half, as enunciated by Britney Gallivan:

1. A single rectangular sheet of paper of any size and uniform thickness can be used.
2. The fold has to be in the same direction each time. (Hence the fold lines are all parallel to each other.)
3. The folding process must not tear the paper. (That is, it must not introduce any discontinuities.)
4. When folded in half, the portions of the inner layers which face one another must almost touch one another.
5. The average thickness or structure of material of paper must remain unaffected by the folding process.
6. A fold is considered complete if portions of all layers lie in one straight line (called *folded section*).

Keywords: *Gallivan, paper folding, half fold, geometric progression, creep section, radius section, series summation, quadratic equation*

Following the rules, we claim that: *The length of the given sheet of paper decides the number of times we can fold it in half*. Thus if we have a sheet of paper of given length, we can calculate the number of times we can fold it theoretically (allowing a reasonable amount of manpower and time).

HOW TO REACH THE SUN …ON A PIECE OF PAPER

A poem by Wes Magee

Take a sheet of paper and fold it,
and fold it again,
and again, and again.
By the 6th fold it will be 1-centimeter thick.
By the 11th fold it will be
32-centimeter thick,
and by the 15th fold - 5-meters.
At the 20th fold it measures 160-meters.
At the 24th fold - 2.5-kilometers,
and by fold 30 it is 160-kilometers high.
At the 35th fold it is 5000-kilometers.
At the 43rd fold it will reach the moon.
And by the fold 52
will stretch from here
to the sun!
Take a piece of paper.
Go on.
TRY IT!

Absolute folding limit

Geometric series. There is a short poem by Wes Magee titled "*How to reach the sun …on a piece of paper*" ([2], page 19), which illustrates the geometric series involved in folding paper in half. After 52 folds (if possible), the width of the folded paper will be approximately equal to the distance between the sun and the earth!

Every time we fold the paper in half, we double the number of layers involved. We have to fold 2^{n-1} sheets of paper for the n^{th} fold. Thus for each successive fold we need more and more energy. Initially this was thought to be the reason for our inability to fold a piece of paper more than 8 times. But, as stated earlier, the strength of the arm is not the limiting factor for the number of folds.

Understanding folds. After each fold, some part of the middle section of the previous layer becomes a rounded edge. The radius of the rounded portion is one half of the total thickness of folded paper; see Figure 1.

Initially the radius is small as compared with the length of the remaining part of sheet. As the folds begin nearing their final thickness, the curved portion becomes more prominent and begins taking up a greater percentage of the volume of the paper. The radius section is the part of the paper 'wasted' in connecting the layers.

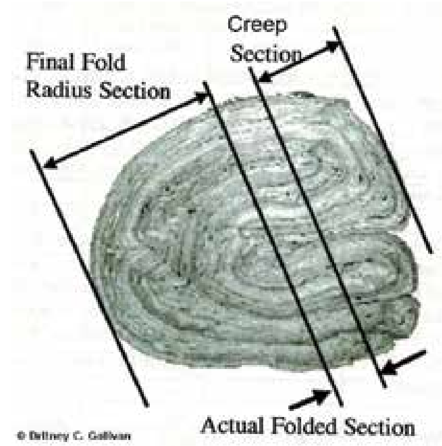

Figure 1. Paper folded in half 12 times illustrating the decrease in folded section and increase in radius section caused due to continued folding which leads to fold losses [©Britney C. Gallivan]; source: [1], [4]

The section that projects past the folded section on the side opposite the radius section is called the *creep* (Figure 1). It is caused by the difference in lengths of layers due to the rounded section of the fold layers having different radii and circumferences.

The limit to the number of folds is reached when a fold has been completed but there is not enough volume or length in the folded section of the paper to fill the entire volume needed for the radius section of the next fold. Thus while making folds there is loss of paper in the form of *radius section* and *creep section*.

Limit formula. Since the *radius* and *creep* sections are semicircular, the length to height ratio of the paper being folded has to be greater than π to allow one more successful fold to occur. *If a folded section's length is less than π times the height, the next fold cannot be completed.*

Let t be the thickness of a sheet of paper. On the first fold, we lose a semicircle of radius t, so the length lost is πt ('lost' in the sense of 'not contributing to the length'). On the second fold, we lose a semicircle of radius t and another semicircle of radius $2t$, so the length lost is

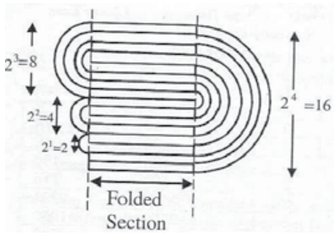

Figure 2. The folded portion and the circular portions [©Britney C. Gallivan]; source: [1], [4]

$\pi t + 2\pi t$ (see Figure 2). Similarly, on the third fold, the length lost is $\pi t + 2\pi t + 3\pi t + 4\pi t$. Generalising this argument, on the n^{th} fold, the length of paper lost is $\pi t + 2\pi t + 3\pi t + \cdots + 2^{n-1}\pi t$.

If n folds are required, the total length lost (L) is:

$$L = \pi t + (\pi t + 2\pi t) + (\pi t + 2\pi t + 3\pi t + 4\pi t) + \cdots + (\pi t + 2\pi t + 3\pi t + \cdots + 2^{n-1}\pi t). \quad (1)$$

Thus the cumulative losses or *minimum length of paper required*, L, of each and every layer and fold can be described by the following equation:

$$L = \sum_{i=1}^{n}\sum_{k=1}^{2^{i-1}} \left(2^{i-1}\pi t - (k-1)\pi t\right)$$

$$= \pi t \sum_{i=1}^{n}\sum_{k=1}^{2^{i-1}} \left(2^{i-1} - (k-1)\right). \quad (2)$$

The first summation, from $i = 1$ to $i = n$, is the summation that increments each physical fold. The second summation, from $k = 1$ to $k = 2^{i-1}$, is a term to allow for the amount of loss of each sheet in a particular fold. Here, the upper limit of 2^{i-1} is one half of the thickness of that fold, or the radius of the outer folded layer of the fold. Since the second summation must end at $2^{i-1}\pi t$ (corresponding to the i^{th} fold), we introduce another variable, k, to insert $(2^{i-1} - 1)$ steps of size πt before the final step. The $(k - 1)$ term steps between the thickness of each sheet in a fold by the sheet's thickness, t. Thus, the above formula computes the radius of each fold and then the length of each sheet in the semicircular fold.

In equations (1) and (2), we calculate the length lost in the i^{th} fold by doing a summation in a "climbing a ladder" fashion and "descending a ladder" fashion (by using the variable k), respectively.

We can simplify the above double summation formula as follows:

$$L = \pi t \sum_{i=1}^{n}\left(\sum_{k=1}^{2^{i-1}} 2^{i-1} - \sum_{k=1}^{2^{i-1}} k + \sum_{k=1}^{2^{i-1}} 1\right)$$

$$= \pi t \sum_{i=1}^{n}\left(2^{2(i-1)} - \frac{2^{i-1}(2^{i-1}+1)}{2} + 2^{i-1}\right)$$

$$= \pi t \sum_{i=1}^{n}\left(\frac{2^{2i-2} + 2^{i-1}}{2}\right)$$

$$= \frac{\pi t}{8}\left(\sum_{i=1}^{n} 2^{2i} + 2\sum_{i=1}^{n} 2^{i}\right)$$

We know the following results:

$$\sum_{i=1}^{m} a^i = a\left(\frac{a^m - 1}{a - 1}\right) \quad \text{and}$$

$$\sum_{i=1}^{m} a^{2i} = \frac{a^2}{a+1}\left(\frac{a^{2m} - 1}{a - 1}\right)$$

Here we have $a = 2$. Hence:

$$L = \frac{\pi t}{8}\left(\frac{4}{3}\left(2^{2n} - 1\right) + 4\left(2^n - 1\right)\right)$$

$$= \frac{\pi t}{2}\left(\frac{2^{2n} - 1}{3} + 2^n - 1\right)$$

$$= \frac{\pi t}{6}\left(2^{2n} + 3\cdot 2^n - 4\right)$$

$$\therefore L = \frac{\pi t}{6}\left(2^n + 4\right)\left(2^n - 1\right).$$

Observe that L is expressed in terms of a quadratic equation in 2^n. Britney found it interesting to realise that when we fold a piece of paper, we are actually finding a solution to a quadratic equation!

Realising the theoretical limit

On 27th January 2002, after eight hours of hard work by three people, a tissue paper (4000 feet long, 0.0033 inches thick) was folded twelve times in half for the first time in Pomona, California. (See Figure 3.) This showed that the commonly held belief that it is not possible to fold a piece of paper in half more than eight times was false!

But note that from the above formula, we can say that approximately 2417 feet of tissue paper (of

Figure 3. Britney Gallivan holding the first sheet of paper ever to be folded twelve times [©Britney C. Gallivan]; source: [1], [4]

thickness 0.0033 inch) would have been enough to accomplish the task of folding paper in half twelve times. For a nice writeup on this episode, please see [5]. See also [6].

Exercise

After the above analysis, the following question may arise: "How many times can we fold a sheet of paper in half, by folding in alternate directions, keeping all other rules the same?"

As expected, the answer is: *The length and width of the given sheet decide the number of times we can fold the paper in half*.

A good idea to proceed will be to start with a square sheet of paper and calculate its limiting width. Without considering the effects of material lost in radii of earlier folds, we get a crude bound for the width W of a square sheet of paper of thickness t required for folding in half n times as:

$$W = \pi t 2^{3(n-1)/2}.$$

I invite readers to derive the above formula.

Surely, the above formula does not give any minimum limit. Using analysis seen in *one direction folding*, we can derive a *Limit Formula* for *alternate folding*. But in *alternate folding*, we get separate equations for odd and even folds. Note that odd folds accumulate losses in an odd fold direction, and even folds accumulate losses in an even fold direction. Also, each fold in an odd direction doubles the amount of paper for the next even direction, and vice versa.

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