Space Filling Curves

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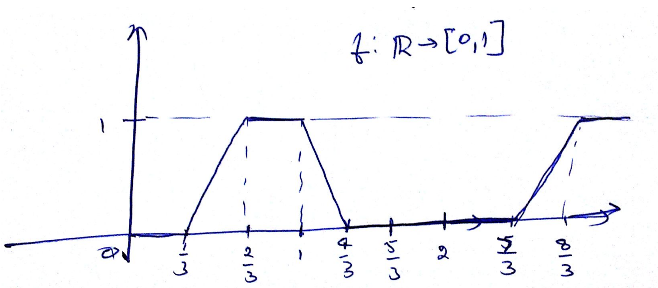
Prove that there exist a continuous surjective map from $\mathbb{R}$ to $\mathbb{R}^2$. In other words, prove that Hilbert's Curve is a continuous surjective map.

Proof. Consider $f : \mathbb{R} \to [0, 1]$



Observe that f is a continuous periodic function with period 2. Hence, $f(x + 2I) = f(x)$ for any integer I.

Now consider, $g : [0, 1] \to [0, 1] \times [0, 1]$ where $g(t) = (x(t), y(t))$. We define the map g as:

$$\begin{cases} x(t) = \frac{1}{2}f(t) + \frac{1}{2^2}f(3^2t) + \frac{1}{2^3}f(3^4t) + \ldots = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f\left(3^{2n}t\right) \\ y(t) = \frac{1}{2}f(3t) + \frac{1}{2^2}f(3^3t) + \frac{1}{2^5}f(3^4t) + \ldots = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f\left(3^{2n+1}t\right) \end{cases}$$

Now this map is continuous from *Weierstrass M-test*, which states that:

> Suppose that $\{f_n\}$ is a sequence of real or complex-valued functions defined on a set A, and that there is a sequence of positive numbers $\{M_n\}$ satisfying
>
> $$\forall n \geq 1, \forall x \in A : \ |f_n(x)| \leq M_n$$
>
> $$\sum_{n=1}^{\infty} M_n < \infty$$
>
> Then the series
>
> $$\sum_{n=1}^{\infty} f_n(x)$$
>
> converges uniformly on A.

Thus, g **is continuous** because f is continuous.

Now we will prove that g is surjective, that is, for all (x_0, y_0) in $[0, 1] \times [0, 1]$ there exist t_0 in $[0, 1]$ such that $x(t_0) = x_0$ and $y(t_0) = y_0$.

> Trick: Represent the elements t_0 in ternary and x_0, y_0 in binary.

*Proof explained by Prof. Manas Ranjan Sahoo, typed by Mr. Gaurish Korpal

Hence we can write:

$$\begin{cases} x_0 = 0.a_0a_2a_4a_6\ldots = \displaystyle\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} a_{2i} \\ y_0 = 0.a_1a_3a_5a_7\ldots = \displaystyle\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} a_{2i+1} \end{cases}$$

where $a_i \in \{0, 1\}$ for all i. Also,

$$t_0 = 0.(2a_0)(2a_1)(2a_2)\ldots = \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} 2a_i$$

where $(2a_i) \in \{0, 2\}$ for all i (motivating the idea of representing t_0 in ternary)

Claim: $x(t_0) = x_0$ and $y(t_0) = y_0$

We will use the ternary expansion of t_0 to simplify,

$$\begin{aligned} 3^n t_0 &= 3^n \left(\sum_{i=0}^{\infty} \frac{2a_i}{3^{i+1}} \right) \\ &= 3^n \left(\sum_{i=0}^{n-1} \frac{2a_i}{3^{i+1}} + \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}} \right) \\ &= 2I + 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}} \end{aligned}$$

since $3^n \geq 3^{i+1}$ for $0 \leq i \leq n-1$, we conclude that $3^n \displaystyle\sum_{i=0}^{n-1} \frac{2a_i}{3^{i+1}}$ is twice an integer (taking 2 out of summation). Now a_n can be either 0 or 1. We will consider following two cases:

Case 1: $a_n = 0$

$$\begin{aligned} 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}} &= 3^n \sum_{i=n+1}^{\infty} \frac{2a_i}{3^{i+1}} \\ &\leq 3^n \sum_{i=n}^{\infty} \frac{2}{3^{i+1}} \qquad \text{(substitute maximum value of } a_i\text{)} \\ &= \frac{1}{3} \qquad \text{(sum of infinite geometric progression)} \end{aligned}$$

Also observe that if we substitute minimum value of a_i we will get $0 \leq 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}}$, hence

$$0 \leq 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}} \leq \frac{1}{3}$$

$$\Rightarrow f\left(2I + 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}}\right) = 0$$

from the definition of f given in the starting.

Case 2: $a_n = 1$

$$\begin{aligned} 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}} &= \frac{2}{3} + 3^n \sum_{i=n+1}^{\infty} \frac{2a_i}{3^{i+1}} \\ &\leq \frac{2}{3} + 3^n \sum_{i=n}^{\infty} \frac{2}{3^{i+1}} \qquad \text{(substitute maximum value of } a_i\text{)} \\ &= \frac{2}{3} + \frac{1}{3} \qquad \text{(sum of infinite geometric progression)} \\ &= 1 \end{aligned}$$

Also observe that if we substitute minimum value of a_i we will get $\frac{2}{3} + 0 \leq 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}}$, hence

$$\frac{2}{3} \leq 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}} \leq 1$$

$$\Rightarrow f\left(2I + 3^n \sum_{i=n}^{\infty} \frac{2a_i}{3^{i+1}}\right) = 1$$

from the definition of f given in the starting.

Now combining both the above cases we conclude that:

$$f(3^n t_0) = a_n$$

Hence:

$$\begin{cases} x(t_0) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f\left(3^{2n} t_0\right) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} a_{2n} = x_0 \\ y(t_0) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f\left(3^{2n+1} t_0\right) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} a_{2n+1} = y_0 \end{cases}$$

Thus proving our claim. Hence we can conclude that g **is a surjective**.

From standard exercises in metric spaces we know that $[0,1]$ is homeomorphic to $\mathbb{R}$ and $[0,1] \times [0,1]$ is homeomorphic to $\mathbb{R}^2$, where *homeomorphism* is defined as

> A function $F : X \to Y$, between two topological spaces[a] (X, T_X) and (Y, T_Y) is called a homeomorphism if it has the following properties:
>
> - F is a bijection,
> - F is continuous,
> - the inverse function F^{-1} is continuous.
>
> If such a function exists, we say X and Y are *homeomorphic*.
>
> [a]it's a set with some special properties, for its definition see http://mathworld.wolfram.com/TopologicalSpace.html

Let $\phi : \mathbb{R} \to [0,1]$ and $\psi : [0,1] \times [0,1] \to \mathbb{R}^2$ be the homeomorphisms, then

$$\begin{tikzcd} {[0,1]} \arrow[r, "g", "\substack{\text{Surjective} \\ \text{Continuous}}"'] & {[0,1] \times [0,1]} \arrow[d, "\psi", "\substack{\text{bijective} \\ \text{continuous}}"'] \\ \mathbb{R} \arrow[u, "\varphi", "\substack{\text{bijective} \\ \text{continuous}}"'] & \mathbb{R}^2 \end{tikzcd}$$

Thus the composition, $\Gamma = \psi \circ g \circ \phi$ is the required **continuous surjective map**.

> **Exercise:** Verify that the function $\Gamma : \mathbb{R} \to \mathbb{R}^2$ defined above will give Hilbert's Curve.

$\square$